Traveling Wave Analysis of Partial Differential Equations

Numerical and Analytical Methods with Matlab and Maple

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(Academic Press, ISBN: 9780123846525)

Chapter Abstracts

Chapter 1
Introduction to Traveling Wave Analysis

Partial differential equations (PDEs) are a general starting point for mathematical modeling and computer-based analysis throughout all of science, engineering and applied mathematics. Computer-based methods for the numerical and analytical solution of PDEs are therefore of broad interest. In this chapter, we discuss some of the general approaches to the traveling wave solution of PDEs, including the method of lines (MOL) for numerical solutions and several approaches such as the tanh, exp and Riccati methods for analytical solutions. Each chapter typically consists of a statement of the PDE system, including areas of applications, a Matlab code for a MOL solution and a Maple code for an analytical solution, each discussed in detail along with the numerical and graphical output. A concluding summary points out the features of the numerical and analytical approaches and how they might be extended to other PDEs. All the computer code discussed in the text is available for download from the website: `www.pdecomp.net`.

Keywords: Partial differential equations; numerical integration; analytical solutions; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 2
Linear Advection Equation

The partial differential equation (PDE) analysis of convective systems is particularly challenging since convective (hyperbolic) PDEs can propagate steep fronts and even discontinuities. To demonstrate this characteristic, we consider in this chapter the numerical and analytical integration of the linear advection equation, possibly the simplest PDE, but ironically, one of the most difficult to integrate numerically. The propagation of moving fronts is illustrated for several cases, from a smooth Gaussian pulse to a discontinuity; the latter is resolved with flux limiters. Matlab code for various MOL solutions are discussed in detail along with the associated numerical and graphical output. The concluding appendix provides a brief introduction to the theory of flux limiters and a survey of fifteen of these mathematical devices which have been used in a spectrum of convective system applications. All the associated Matlab code is available for download.

Keywords: Linear advection equation; smoothness of solutions; propagation of discontinuities; method of lines; MOL; numerical integration; flux limiters; traveling waves; Matlab

Chapter 3
Linear Diffusion Equation

The one-dimensional (1D) diffusion equation, also termed *Fourier's second law* or *Fick's second law* is a basic parabolic partial differential equation (PDE) that admits traveling wave solutions. We first demonstrate how an assumed Lagrangian change of variable transforms the PDE to an ordinary differential equation (ODE) that can be integrated analytically; the ODE solution is then transformed back to a traveling wave solution of the PDE. This analytical solution is used to evaluate a numerical solution computed by the method of lines (MOL). The Matlab routines for the MOL solution are discussed in detail, and the numerical and graphical output from the routines is presented. All the associated computer code is available for download.

Keywords: 1D linear diffusion equation; parabolic PDE; traveling wave solutions; method of lines; MOL; numerical integration; Matlab

Chapter 4
A Linear Convection Diffusion Reaction Equation

In this chapter, the one-dimensional (1D) advection (hyperbolic) equation of Chapter 2 and the 1D diffusion (parabolic) equation of Chapter 3 are combined into a partial differential equation (PDE) that also includes a first-order reaction term. This convection-diffusion-reaction (CDR) PDE, which can be termed a mixed hyperbolic-parabolic PDE, is integrated numerically and analytically, and the two solutions are compared. The numerical solution is obtained by the method of lines (MOL) with a detailed discussion of the Matlab MOL routines. The analytical solution is achieved by traveling wave analysis, starting with a Lagrangian change of variable that transforms the CDR PDE into an ODE; the analytical solution of the ODE is then changed back to a solution for the PDE. One aspect of this analytical solution is the use of a boundary condition (BC) at infinity. This also requires the selection of a distance scale in the numerical solution that is effectively infinite. All the associated computer code is available for download.

Keywords: convection-diffusion-reaction PDE; hyperbolic-parabolic PDE; traveling wave; method of lines; MOL; numerical integration; analytical solution; infinite BC; Matlab

Chapter 5
Diffusion Equation with Nonlinear Source Terms

The diffusion equation of Chapter 3 is extended to include a second-order and a third-order source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. An analytical solution is derived by the factorization method as outlined in the main Appendix. The factorization method is used to derive two different traveling wave solutions, one of which is the same as that used to validate the numerical solution. Maple code is presented that performs this procedure automatically. All the associated comuter code is available for download, including additional Maple code that solves the PDE problem using the exp, tanh and Riccati methods.

Keywords: Partial differential equations; diffusion equation with nonlinear source terms;numerical integration; analytical solutions; traveling wave analysis; method of lines; MOL; factorization; exp, tanh, Riccati methods; Matlab; Maple

Chapter 6
Burgers-Huxley Equation

The Burgers-Huxley partial differential equation (PDE) is an extension of the diffusion equation of Chapter 3 with a nonlinear convection term and a third and fifth order source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. An analytical solution to the generalized Burgers-Huxley PDE is derived by the factorization method as outlined in the main Appendix. Maple code is presented which performs this procedure automatically, and then assigns values to constants in order to obtain the specific solution used to validate the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp, tanh and Riccati methods.

Keywords: Partial differential equations; Burgers-Huxley equation; analytical solutions; traveling waves; method of lines; MOL; numerical integration; factorization; exp, tanh, Riccati methods; Matlab; Maple

Chapter 7
Burgers-Fisher Equation

The Burgers-Fisher partial differential equation (PDE) is an extension of the diffusion equation of Chapter 3 with a nonlinear convection term and a first and third order source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. The appendix of this chapter illustrates the application of a mathematical transformation to the generalized Burgers-Fisher Equation (discussed in Chapter 6) to convert it to a new simpler form. Then the tanh method, as outlined in the main Appendix, is used to obtain traveling wave solutions. Maple code is presented which performs this procedure automatically, and then assigns values to constants in order to obtain the specific solution used for the validate the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and Riccati methods.

Keywords: Partial differential equations; generalized Burgers-Fisher equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 8
Fisher-Kolmogorov Equation

The Fisher-Kolmogorov partial differential equation (PDE) is an extension of the diffusion equation of Chapter 3 with a linear source term and a source term of arbitrary order (the order is a parameter). A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. The appendix to this chapter illustrates how the factorization method, as outlined in the main Appendix, can be used to obtain traveling wave solutions to the Fisher-Kolmogorov equation. Maple code is presented that performs this procedure automatically, and then assigns values to constants in order to obtain the specific solution used for the numerical analysis. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp, tanh and Riccati methods.

Keywords: Partial differential equations; Fisher-Kolmogorov equation; analytical solutions;numerical integration; traveling waves; method of lines; MOL; factorization; exp, tanh, Riccati methods; Matlab; Maple

Chapter 9
Fitzhugh-Nagumo Equation

The Fitzhugh-Nagumo (F-N) partial differential equation (PDE) is an extension of the diffusion equation of Chapter 3 with a linear and a cubic source term. The BCs include a single pulse and a train of pulses in time. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. The appendix to this chapter analyses the spatially distributed, coupled Fitzhugh-Nagumo equations, specifically, how the application of an external stimulus to an axon can result in a traveling wave along an excitable medium (e.g., nerve). In addition, the tanh and exp methods, as outlined in the main Appendix, are used to obtain traveling wave solutions to the single equation form of the F-N equations. Maple code is presented which performs this procedure automatically to obtain the specific solution used to evaluate the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the Riccati method.

Keywords: Partial differential equations; Fitzhugh-Nagumo equation;numerical integration; analytical solutions; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 10
Kolmogorov-Petrovskii-Piskunov Equation

The Kolmogorov-Petrovskii-Piskunov equation, also called the Fisher-KPP equation or just the KPP equation, is a one-dimensional (1D) diffusion equation with a linear source term and a source term of arbitrary order (the order is a parameter). A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. An analytical solution to the KPP equation is derived by the factorization method, as outlined in the main Appendix. Maple code is presented that performs this procedure automatically to obtain the specific analytical solution used to validate the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp, tanh and Riccati methods.

Keywords: Partial differential equations; Kolmogorov-Petrovskii-Piskunov equation; KPP equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; factorization; exp, tanh, Riccati methods; Matlab; Maple

Chapter 11
Kuramoto-Sivashinsky Equation

The Kuramoto-Sivashinsky equation has a nonlinear convection term and second, third and fourth order spatial (boundary value) derivatives. Thus, this example illustrates the solution of a higher (fourth) order PDE. The four required boundary conditions (BCs) are taken from the analytical solution as two Dirichlet BCs and two Neumann BCs. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the chapter appendix, diffusion-induced chaos produced by the Kuramoto-Sivashinsky equation is discussed whereby a simple initial condition can lead to chaotic behavior. Additionally, traveling wave solutions are derived using the tanh method, as outlined in the main Appendix. Maple code is presented which performs this procedure automatically to obtain the specific analytical solution used to validate the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and Riccati methods.

Keywords: Partial differential equations; Kuramoto-Sivashinsky equation; diffusion-induced chaos; analytical solutions; numerical integration; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 12
Kawahara Equation

The Kawahara equation has a nonlinear convection term, and a third and a fifth order spatial (boundary value) derivative. Thus, this example illustrates the solution of a higher (fifth) order PDE. Although technically five boundary conditions (BCs) are required, the spatial domain is specified to be long enough that the solution and its derivatives do not depart from the initial zero values. Therefore, only two Dirichlet BCs from the analytical solution are used and derivative BCs are not used; the Dirichlet BCs are subsequently dropped and the solution does not change. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In Appendix 2 to this chapter the dispersion relation of the Kawahara equation is analysed, leading to the conclusion that the system is purely dispersive and that, under certain conditions, the dispersion is canceled by nonlinear effects. Maple code is presented which performs the analysis automatically. All the associated computer code is available for download, including additional Maple code that derives traveling wave solutions using the exp, tanh and Riccati methods.

Keywords: Partial differential equations; Kawahara equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; dispersion relation; exp, tanh, Riccati methods; Matlab; Maple

Chapter 13
Regularized Long-Wave Equation

The regularized long-wave (RLW) equation has a linear and a nonlinear convection term, and a mixed partial derivative, first order in the initial value variable and second order in the spatial (boundary value) variable. A method of lines (MOL) analysis of the PDE leads to a system of coupled ODEs for which a numerical solution consists first of uncoupling the ODEs algebraically followed by conventional numerical integration of uncoupled ODEs. The analysis of the MOL solution includes a detailed discussion of the Matlab routines and the numerical and graphical output. The appendix to this chapter discusses some of the background to the RLW equation, and includes traveling wave solutions using the exp, tanh and Riccati methods. In addition, analytical traveling wave solutions for coupled RLW equations are obtained using the tanh method. All the associated compter code is available for download.

Keywords: Partial differential equations; regularized long-wave equation; RWL; analytical solutions; numerical integration; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 14
Extended Bernoulli Equation

In the preceding chapters, traveling wave analysis took the form of applying a Lagrangian change of variable to a PDE to give an ODE that is easier to solve analytically, particularly if a computer algebra system such as Maple is applied to the ODE. An analytical solution to the PDE is then produced by applying the inverse of the original transformation to the ODE. In this chapter the process is reversed by starting with an ODE, the Bernoulli ODE; an analytical solution of the ODE is then converted to the solution of a PDE by using an inverse Lagrangian transformation. The PDE, which can be considered as an extension of the Bernoulli ODE, is a variation of the linear advection equation with a linear and a quadratic source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. The appendix to this chapter includes an analytical solution of the extended Bernoulli equation produced from an application of the tanh method. Maple code is presented that performs this procedure automatically to obtain the specific solution used for the evaluation of the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and Riccati methods.

Keywords: Partial differential equations; extended Bernoulli equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 15
Hyperbolic Liouville Equation

The PDEs discussed in the preceding chapters were first order in the initial value variable. This chapter is the first pertaining to a PDE second order in the initial value variable, the hyperbolic Liouville equation, which has a second order derivative in the spatial (boundary value) variable and an exponential source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the appendix to this chapter, two different mathematical transformations are applied to the hyperbolic Liouville equation in order to convert it to new simpler forms. The first form leads to a rational solution by the variables separable method, whilst the second form yields a traveling wave solution by application of the Riccati method, as outlined in the main Appendix. Maple code is presented which performs these procedures automatically. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and tanh methods.

Keywords: Partial differential equations; hyperbolic Liouville equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL;exp, tanh, Riccati methods; Matlab; Maple

Chapter 16
Sine-Gordon Equation

The sine-Gordon equation is the classical wave equation with a nonlinear sine source term. A numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the appendix to this chapter, a series of mathematical transformations is applied to the sine-Gordon equation in order to convert it to a form that can be solved. The new form appears to be considerably more complicated than the original; however, it readily yields a traveling wave solution by application of the tanh method, as outlined in the main Appendix. Interestingly, the solution is a peakon (peaked soliton). Maple code is presented that performs these procedures automatically. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and Riccati methods.

Keywords: Partial differential equations; sine-Gordon equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; peakon; exp, tanh, Riccati methods; Matlab; Maple

Chapter 17
Mth-Order Klein-Gordon Equation

The mth-order Klein-Gordon equation has a second order derivative in the initial and spatial (boundary value) variables, a linear source term and an mth-order source term. When m is two, this is the quadratic Klein-Gordon equation; when m is three, it is the cubic Klein-Gordon equation. When both source terms are dropped, it is the linear wave equation as a special case. The numerical solutions are computed for both the nonlinear case (with source terms) and the linear case (no source terms); for the later, the classical d'Alembert solution is produced. The numerical solutions are computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the appendix to this chapter, traveling wave solutions to the mth-order Klein-Gordon equation are derived by application of the tanh method. Maple code is presented that performs this procedure automatically to obtain the specific solution used for the validation of the numerical solution. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and Riccati methods.

Keywords: Partial differential equations; mth-order Klein-Gordon equation; analytical solutions; numerical integration; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Chapter 18
Boussinesq Equation

The Boussinesq equation discussed in this chapter has: (1) a second order derivative in the initial value variable, (2) second order derivatives in the spatial (boundary value) variable with respect to the dependent variable and the square of the dependent variable, and (3) a fourth order derivative in the spatial variable. The four required boundary conditions (BCs) are homogeneous in the dependent variable (Dirichlet BCs) and the second derivative of the dependent variable. The fourth order spatial derivative is calculated by a finite difference (FD) derived specifically for fourth derivatives and by application of stage-wise differentiation in which a FD for second derivatives is used twice. The numerical solutions are computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the appendix to this chapter, the background to the Boussinesq equation and conditions that must be satisfied for solitary wave solutions to exist are presented. Solutions by application of direct integration and Riccati methods are obtained that match the solution used for verification of the numerical solutions. Maple code for the Riccati based solution is presented. All the associated computer code is available for download, including additional Maple code that solves the PDE problem using the exp and tanh methods.

Keywords: Partial differential equations; Boussinesq equation; numerical integration; analytical solutions; traveling waves; method of lines; MOL; direct integration; exp, tanh, Riccati methods; Matlab; Maple

Chapter 19
Modified Wave Equation

In this chapter, the procedure discussed in Chapter 14 of starting with an ODE, then using an inverse Lagrangian transformation to obtain a PDE is applied to an ODE that leads to a PDE second order in the initial value variable (rather than first order as in Chapter 14). The resulting PDE has the form of the linear wave equation, so it is termed a modified wave equation. The numerical solution is computed by the method of lines (MOL), including detailed discussion of the Matlab routines and the numerical and graphical output. In the appendix to this chapter, a traveling wave solution to the modified wave equation is derived by application of the Riccati method. Maple code is presented that performs this procedure automatically to obtain the specific solution used for the numerical analysis. All the associated compter code is available for download, including additional Maple code that solves the PDE problem using the exp and tanh methods.

Keywords: Partial differential equations; modified wave equation; numerical integration; analytical solution; traveling waves; method of lines; MOL; exp, tanh, Riccati methods; Matlab; Maple

Appendix A
Analytical Solution Methods for Traveling Wave Problems

This appendix outlines a number of methods for obtaining traveling wave solutions to PDEs. These methods, which are currently active areas of research, are used throughout the book to obtain analytical solutions to PDE problems that are then used to validate the numerical solutions obtained by application of the method of lines (MOL). The tanh, exp and Riccati expansion methods are discussed in detail, along with factorization and direct integration methods. Each method is presented along with illustrative examples and Maple code that obtains the solutions automatically. The Maple analysis for the expansion methods is then extended in order to develop stand alone procedures. The emphasis is primarily on 1D PDE models. However, the tanh method is developed further in order to solve 1D coupled and 2D PDE problems. Examples are provided for each method which demonstrate how these procedures can be applied to solve a wide range of PDE traveling wave problems. All the associated Maple code is available for download.

Keywords: Partial differential equations; modified wave equation; numerical integration; analytical solutions; traveling waves; method of lines; MOL; direct integration; factorization; expansion methods; exp, tanh, Riccati methods; Maple